The Minimum Spanning Tree Problem

Given a connected graph, find a spanning tree of minimum total edge cost.

where,

n = the number of vertices

m = the number of edges

$$n-1 \leq m \leq \binom{n}{2}$$

Applications

Network Construction

Clustering

Minimum Tour Relaxation (Held-Karp 1-trees)

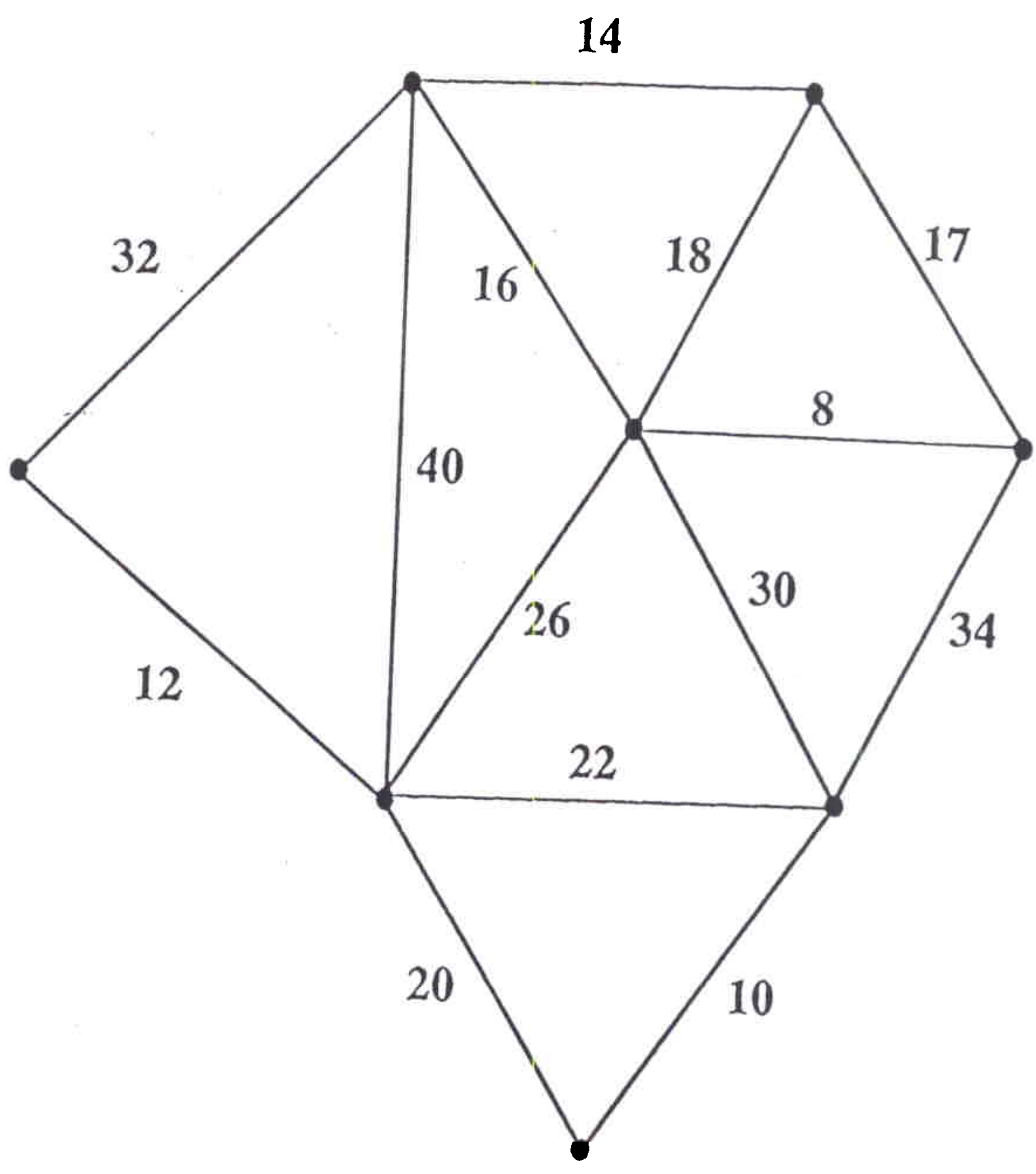
A Simple Solution From the 80's

(with apologies to Oliver Stone)

Gorden Gecko: "Greed is Good"

Repeatedly select the cheapest unselected edge and add it to the tree under construction if it connects two previously disconnected pieces.

Kruskal, 1956


14
32
16
18
17
8
40
26
30
34
12
22
20
10

The greedy method generalizes to matroids.

We shall generalize the method rather than the domain of application.

Generalized Greedy Method

Beginning with all edges uncolored,

sequentially color edges

blue (accepted) or red (rejected).

Blue Rule:

> **Color blue any minimum-cost uncolored edge crossing a cut with no blue edges crossing.**

Red Rule:

> **Color red any maximum-cost uncolored edge on a cycle with no red edges.**

∃ MST with all blue, no red

Jarnik's Algorithm

Grow a tree from a single start vertex.

At each step add a cheapest edge with exactly one end in the tree.

Boruvka's Algorithm

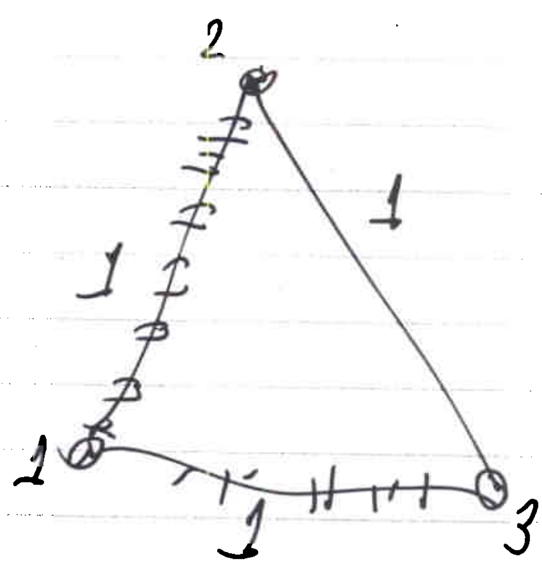
Repeat the following step until
all vertices are connected:

> For each blue component, select a cheapest edge connecting to another component; color all selected edges blue.

For correctness, a tie-breaking rule is needed.

Henceforth, assume all edge costs are distinct.

Then there is a unique spanning tree.



2
1
1
1
1
3

"Classical" Algorithms

(before algorithm analysis)

Kruskal's algorithm, 1956

$O(m \log n)$ **time**

Jarnik's algorithm, 1930

$O(n^2)$ **time**

also Prim, Dijkstra

Boruvka's algorithm, 1926

$O(\min\{m \log n, n^2\})$ **time**

and many others

Selected History

Boruvka, 1926	$O(\min\{m\log n, n^2\})$
Jarnik, 1930 **Prim, 1957** **Dijkstra, 1959**	$O(n^2)$
Kruskal, 1956	$O(m\log n)$
Williams, Floyd, 1964 **heaps**	$O(m\log n)$
Yao, 1975 **packets in Boruvka's algorithm**	$O(m\log\log n)$
Fredman, Tarjan, 1984 **F-heaps in:**	
Jarnik's algorithm	$O(n\log n + m)$
a hybrid Jarnik-Boruvka algorithm	$O(m\log^* n)$
Gabow, Galil, Spencer, 1984 **Packets in F-T algorithm**	$O(m\log\log^* n)$

$$\log^* n = \min\{i \mid \log\log\log\ldots\log n \le 1\}$$

where the logarithm is iterated i times

Models of Computation

We assume comparison of the two edge costs takes unit time, and no other manipulation of edge costs is allowed.

Another model:

bit manipulation of the binary representations of edge costs is allowed.

In this model,

Fredman-Willard, 1990, achieved O(m) time.

(fast small heaps by bit manipulation)

Goal: An O(m)-time algorithm

without bit manipulation of edge weights

Boruvka's algorithm with contraction:

If G contains at least two vertices:

select cheapest edge incident to each vertex;

Contract all selected edges;

Recur on contracted graph.

If contraction preserves sparsity (m = O(n)),

this algorithm runs in O(n) = O(m) time

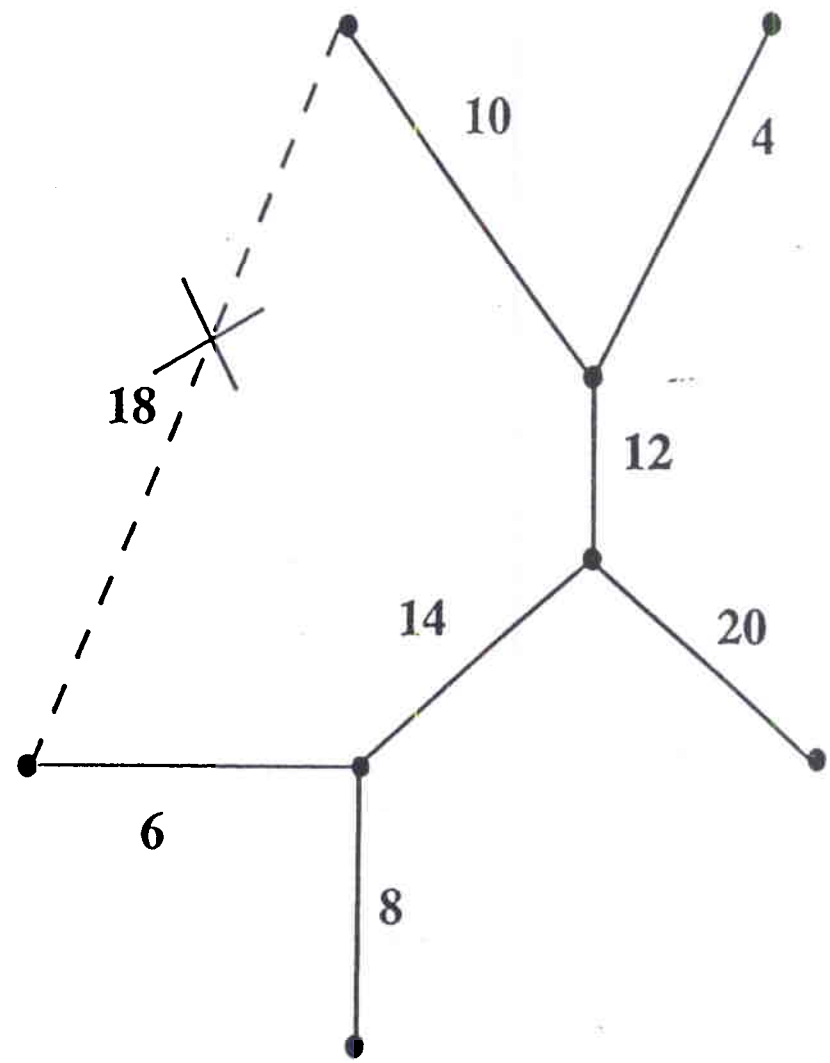
on sparse graphs.

E.g. planar graphs

How to handle non-sparse graphs?

Thinning: remove all but O(n) edges by finding edges that can't be in the minimum spanning tree.

How to thin?



10
4
18
12
14
20
6
8

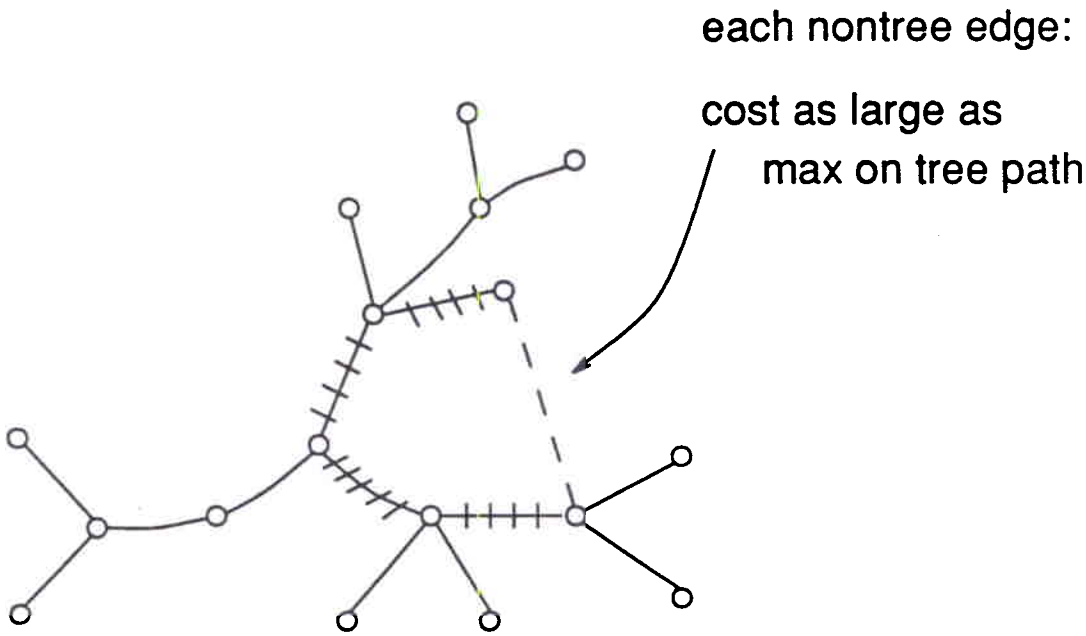
Verification:

Given a spanning tree, is it minimum?

Thinning: Given a spanning tree, delete any non-tree edge larger than every edge on tree path joining its ends (red rule).

If all non-tree edges can be thinned, tree is verified.

Verification



History of Verfication Algorithms

Tarjan, 1979	$O(m\,\alpha\,(m,n))$ **time**
Komlos, 1984	$O(m)$ **comparisons**
Dixon, Rauch, Tarjan, 1992	$O(m)$ **time**
King, 1993	$O(m)$ **time (simplified)**

All these algorithms will thin.

Thinning by Random Sampling (1993)

Select half the edges at random.

Build a minimum spanning forest of the sample.

Thin.

How many edges remain?

Karger:	O(nlogn) on average
Klein, Tarjan:	$<$ 2n on average

Minimum Spanning Forest Algorithm

If # edges/ # vertices < 5, then

(Boruvka step) Select the cheapest edge incident to each vertex.

Contract all selected edges.

Recur on contracted graph.

Else

(Sampling and Thinning Step) Sample the edges, each with probability 1/2.

Construct a minimum spanning forest of the sample, recursively.

Thin using this forest.

Recur on Thinned Graph

Analysis

Boruvka step

$m < 5n$ implies $m' < 9m/10$ since at least

$n/2$ edges are contracted

$T(m) = O(m) + T(9m/10)$

Thinning Step

$m > 5n$ implies $2n < 2m/5$

$T(m) = O(m) + T(m/2) + T(2m/5)$

where $T(m/2)$ and $T(2m/5)$ are expected time

$T(m) = O(m)$ by induction

Bound on Number of Edges Not Thinned

Let $e_1, e_2, \ldots, e_m$ be the edges, in increasing cost.

Run the following variant of Kruskal's algorithm.

Inititalize $F = \varnothing$.

Process the edges in order.

> To process e_i, flip a coin to see if e_i is in the sample.
>
> If e_i forms a cycle with edges in F, discard it as thinned.
>
> Otherwise, if e_i is sampled, add e_i to F. (Whether or not e_i is sampled, it is not thinned.)

F is the minimum spanning forest of the sample.

How many edges are not thinned?

The only relevant coin flips are those on unthinned edges, each of which has a chance of 1/2 of adding an edge to F (a success).

There can be at most n-1 successes.

For there to be more than k unthinned edges, the first k relevant coin flips must give at most n-2 successes.

The chance of this is at most

$$(\frac{1}{2})^k \sum_{i=0}^{n-2} \binom{k}{i} < (\frac{1}{2})^k \sum_{i=0}^{n} \binom{k}{i}$$

In particular, the average number of unthinned edges is at most 2n.

Further Results

$O(m\alpha(n)\log(\alpha(n))) \rightarrow O(m\alpha(n))$ deterministic

Chazelle: "soft" heaps

Optimal to within a constant factor

Pettie + Ramachandran:

Chazelle + optimal on small subproblems

Open Problems

Deterministic O(m)?

Simpler verification?

Other applications?

directed spanning trees?

shortest paths?